The pseudo-analytical method: application of pseudo-Laplacians to acoustic and acoustic anisotropic wave propagation

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Summary

We generalize the pseudo-spectral method for the acoustic wave equation to create analytical solutions to the constant velocity acoustic wave equation in an arbitrary number of space dimensions. We accomplish this by modifying the Fourier Transform of the Laplacian operator so that it compensates exactly for the error due to the second-order finite-difference time marching scheme used in the conventional pseudo-spectral method. Of more practical interest, we show that this modified or pseudo-Laplacian is a smoothly varying function of the parameters of the acoustic wave equation (velocity most importantly) and thus can be further generalized to create near-analytically-accurate solutions for the variable velocity case. We call this new method the pseudo-analytical method. We further show that by applying this approach to the concept of acoustic anisotropic wave propagation, we can create scalar-mode VTI and TTI wave equations that overcome the disadvantages of previously published methods for acoustic anisotropic wave propagation. These methods should be ideal for forward modeling and reverse time migration applications.

Introduction

The pseudo-spectral method (Reshef et al., 1988) is generally considered an accurate method for solving equations such as the acoustic or elastic wave equations. However, it still suffers from errors, namely grid dispersion, due to the fact that second-order (or sometimes higher-order) finite differences are applied on the time axis. Etgen (2007) describes a technique built upon the work of Holberg (1987) that includes the effect of second-order time discretization with finite time-step size to partially compensate errors due to space discretization, thus creating a globally optimized finite-difference scheme. However, optimized finite-difference techniques still must use some degree of oversampling compared to the Nyquist limit.

Any technique that strives for "perfect" accuracy either has to have no error in both the time and space discretizations, or have errors in both that cancel each other exactly. Tal-Ezer et al. (1987) described an approach based on the former by using pseudo-spectral space derivatives coupled with an orthogonal polynomial expansion in time. While this method is accurate, it is somewhat cumbersome to code and has seen little industrial use to our knowledge. Our approach is of the later type; we use our freedom to modify the wavenumber response of the Laplacian, or other spatial derivatives that we need, directly in the wavenumber domain portion of a pseudo-spectral-like method to cancel the error caused by second-order finite-difference time marching.

The pseudo-analytical method

The accuracy of a numerical wave propagation scheme can be determined in the constant wave-speed case by finding the expression of the Fourier Transform of the method and solving that expression for temporal frequency as a function of all the other variables. Equation 1 gives an expression for the Fourier Transform of a second-order time-marching solution to the acoustic wave equation where we've left the details of the spatial difference method generic:

$$(2\cos(\omega\Delta t)-2)\ u(\omega,\vec{k}) = \Delta t^2 v^2 F(\vec{k})\ u(\omega,\vec{k}) \qquad (1)$$

We rearrange this expression to give temporal frequency as a function of velocity, the spatial Fourier Transform of the spatial differential operator and the time step size.

$$\omega = \frac{\cos^{-1}[\ \frac{1}{2}v^2\Delta t^2 F(\vec{k})+1]}{\Delta t} \qquad (2)$$

Then, we compute phase velocity as a function of wavenumber by dividing temporal frequency by the magnitude of the wavenumber vector.

$$v(k)_{phase} = \frac{\cos^{-1}[\ \frac{1}{2}v^2\Delta t^2 F(\vec{k})+1\ ]}{\Delta t\left|\vec{k}\right|} \qquad (3)$$

Written this way, we find the numerical phase velocity of an acoustic wave propagation algorithm once we know the time step size and the spatial Fourier Transform of the implementation of the spatial differential operators. Indeed, if we wished to know the numerical phase velocity for a pseudo-spectral method, we'd simply use $-\left|\vec{k}\right|^2$ for $F(\vec{k})$.

The key innovation is to realize that we can engineer whatever phase velocity we wish simply by rearranging the expression in equation 3 to solve for the Fourier Transform of the spatial differential operators of the wave equation.

$$F(k)=\frac{2\cos[v\ \Delta t\ |\vec{k}|]-2}{v^2\Delta t^2} \quad (4)$$

For the acoustic isotropic wave equation, v is set to a constant, the correct phase velocity that we wish to have the scheme produce. This $F(\vec{k})$ is very similar to, but not exactly equal to the Fourier transform of the Laplacian operator $-|\vec{k}|^2$. In the limit as time step size approaches zero $F(\vec{k})$ would approach the Fourier Transform of the Laplacian. This new operator is no longer a differential operator; it is a pseudo-differential operator.

This result is interesting, in that it gives a recipe for computing the analytical solution to the acoustic wave equation in an arbitrary-dimensional constant velocity periodic domain, (periodic due to the FFT), with a simple second-order time-marching scheme. Figure 1 shows the spatial Fourier Transform of the Laplacian (a) (which would be used in a pseudo-spectral method), the spatial Fourier Transform of the pseudo Laplacian $F(\vec{k})$ for a velocity of 1500m/s (b) and the spatial Fourier Transform of the pseudo Laplacian $F(\vec{k})$ for a velocity of 3300 m/s (c) given a time step size of .001 seconds. The Courant numbers (v dt/min(dx,dz)) for these cases are 0.15 and 0.33 respectively. Observe that the Fourier Transforms of the pseudo-Laplacians differ from the standard Laplacian more as the Courant number increases.

Of course, this result so far isn't very interesting for computing industrial wave simulations since we are usually interested in variable velocity media. The next key realization is that while the pseudo-Laplacian is a function of velocity and grid sizes, it is a slowly varying and regular function of those parameters.

Figure 2 (a) shows the pseudo-Laplacian for an intermediate Courant number = 0.22, created by interpolating the pseudo-Laplacians at 1500 m/s and 3300m/s. Figure 2(b) shows the difference between the "exact" pseudo-Laplacian for this Courant number (using equation 4) and the interpolated one scaled up a factor of 100. The error is essentially trivial.

The fact that the pseudo-Laplacian for a velocity v can be interpolated from pseudo-Laplacians v1< v < v2 in the wavenumber domain leads directly to a straightforward algorithm for accurate time marching in a variable velocity medium. Create pseudo-Laplacians for vmin and vmax. At each time step apply them in the wavenumber domain to the current value of the wavefield. Inverse transform these wavefields back to the space domain creating 2 reference versions of the pseudo-Laplacian applied wavefields. At each point in the computational domain compute the pseudo- Laplacian wavefield appropriate to the local velocity from the 2 reference wavefields. Perform the usual second-order time update.

Figure 3 (a) shows a velocity model with smooth variation and the impulse responses (with magnified insets) of the conventional pseudo-spectral method (b) and the proposed pseudo-analytical algorithm (c). The grid dispersion in figure 3 (b) is purely due to the error in second-order time-differencing that is not compensated because we use the perfect space derivative operator $-|\vec{k}|^2$. Figure 4 (c) is dispersion free since the spatially-varying pseudo-Laplacian compensates, almost exactly, *everywhere*, for the error in caused by second-order time marching.

The potential of this approach is not limited to isotropic acoustic wave propagation alone. Anisotropic P wave propagation is usually accomplished (Alkhalifa, 2000) by taking the elastic wave equation and deriving a differential equation where the pseudo-shear-wave velocity is set to zero for some angle of propagation. The result is a vector system of differential equations that will generate a pseudo-P wave solution that is a reasonable approximation to the P-wave solution in an elastic anisotropic medium (Fletcher et al., 2008). The drawback to this technique is that we have to solve a second-order vector wave equation which must generate an additional solution along with the pseudo P waves we are interested in. These spurious and undesired solutions can lead to noise and artifacts in a wave propagation experiment. This is particularly annoying when we are using the method for reverse time migration.

If we take equation 4, and generalize it to allow velocity to vary with propagation direction (which is directly described in the wavenumber domain) we can derive a scalar second-order in time pseudo-differential equation for anisotropic P wave propagation. Equation 5 gives a simple example of one of the possible expressions for P wave phase velocity in a VTI medium (Harlan and Lazear, 1998):

$$v^2{}_{phase}(\vec{k})=v_v^2k_z^2+(v_n^2-v_h^2)k_x^2k_z^2/k_r^2+v_h^2k_x^2 \quad (5)$$

Plugging this expression into equation 4 leads to a pseudo-analytical wave propagation algorithm for P waves in a

VTI medium parameterized by horizontal , NMO, and vertical velocity.

$$F(\vec{k}) = \frac{2\cos\left[v_{phase}(\vec{k})\ \Delta t\ |\vec{k}|\right] - 2}{v^2 \Delta t^2} \qquad (6)$$

Practically, direct implementation of equation 6 would require pseudo-Laplacians to be created for a range of all 3 velocity parameters and used through interpolation to create the correct operator at each point according to its local velocity values. With 3 velocity parameters and 2 reference velocities each, 8 inverse FFT's would have to be computed at each time step. We can sidestep some of that computational burden by dissecting the required pseudo-Laplacian of equation 6 into 3 constituent parts, the analogues of k_z^2, k_x^2, and $k_z^2 k_x^2 / k_r^2$. Rather than applying the applying the entire pseudo-Laplacian, we apply these 3 operators and then combine them as a function of velocity to create the effect of the variable velocity pseudo-differential operator that propagates anisotropic P waves. Figure 4 shows a wavefront computed in a model like the one in Figure 3, but now for a VTI medium. Note that there are no spurious wave modes.

Finally, the derivation above for scalar VTI anisotropic P waves can be extended to TTI media by rewriting equation 5 recognizing that we need to convert the differential and pseudo-differential operators above into directional differential operators (through vector calculus) along and perpendicular to the symmetry axis. Fortunately, the direction cosines of the symmetry axis factor out of the expression in a way that they are purely scalars applied in the same way that the velocity is applied in the pseudo-spectral method. Figure 5 shows a wavefront in a medium with the same wave speeds as in Figure 4, but with a spatially variable symmetry axis (note the "tilt" in the impulse response).

Conclusions

It is possible to modify the pseudo-spectral method to create a time-marching solution to acoustic and other wave equations that has analytic or near-analytic accuracy. In addition, this concept of a "pseudo-Laplacian" allows the creation of a purely scalar anisotropic wave equation with essentially whatever symmetry one desires.

This technique is effective and efficient any time the pseudo-spectral technique is viable, in particular when one seeks accurate solutions of wave equations on sparse grids.

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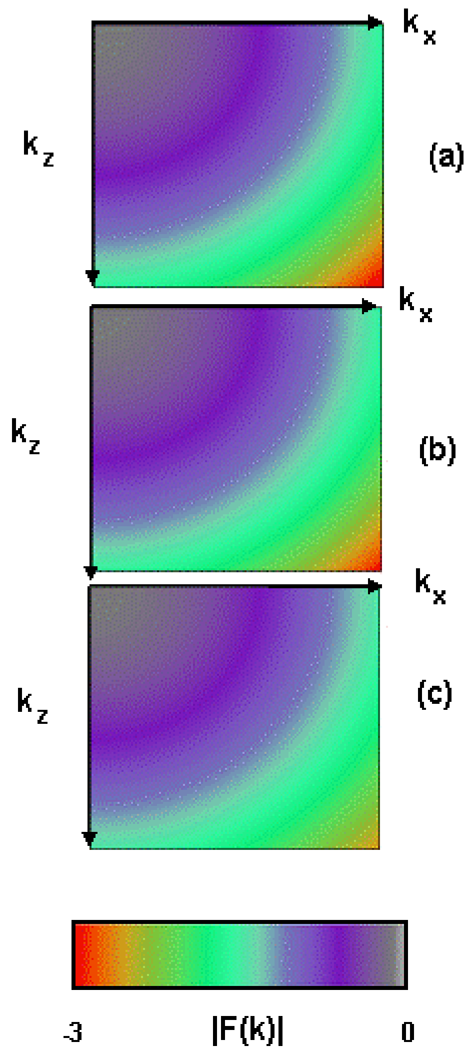

Figure 1: Laplacicans and pseudo-Laplacians for the acoustic wave Equation.

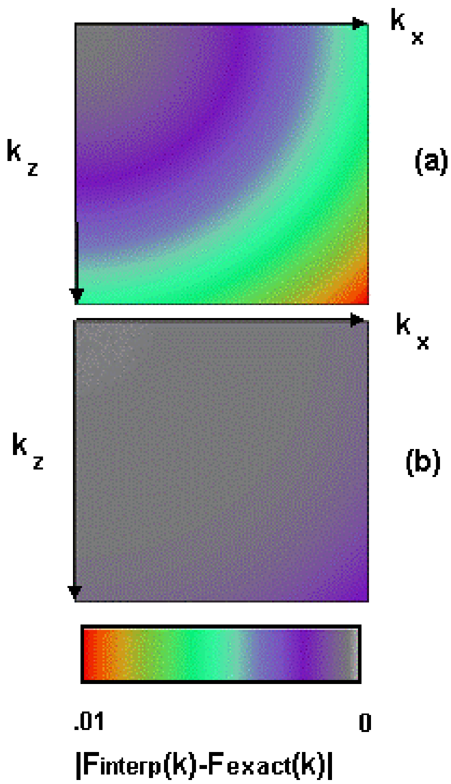

Figure 2: Exact wavenumber pseudo-Laplacian for intermediate velocity (a) and scaled difference between interpolated and exact pseudo-Laplacian (b).

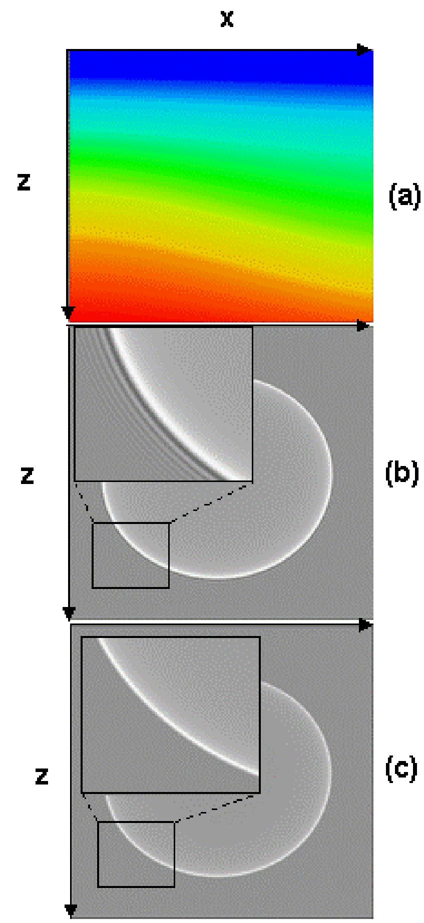

Figure 3: Velocity model (a) and acoustic wavefield snapshots for (b) pseudo-spectral and (c) pseudo-analytical method.

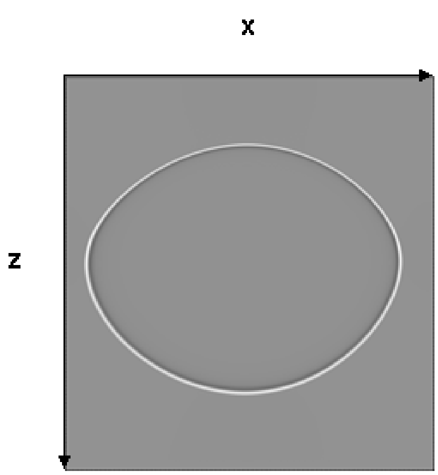

Figure 4: Scalar wave snapshot in a VTI medium.

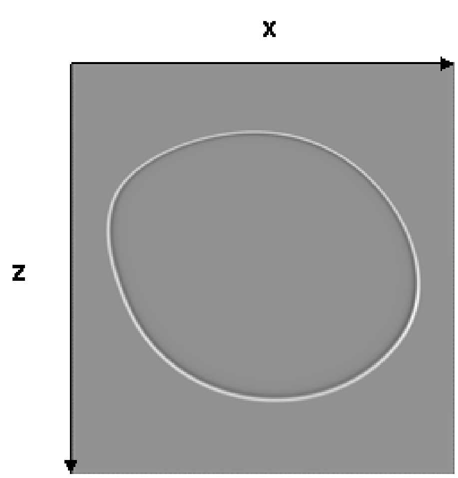

Figure 5: Scalar wave snapshot in a TTI medium.

EDITED REFERENCES

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